

Amplification or reduction of backscattering in a coherently amplifying or absorbing disordered chain

Asok K. Sen*

International Centre for Theoretical Physics

P.O. Box 586, 34100 Trieste, Italy

and

LTP Division, Saha Institute of Nuclear Physics

1/AF, Bidhannagar, Calcutta 700 064, India

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Abstract

We study localization properties of a one-dimensional disordered system characterized by a random non-hermitean hamiltonian where both the randomness and the non-hermiticity arises in the local site-potential; its real part being random, and a constant imaginary part implying the presence of either a coherent absorption or amplification at each site. While the two-probe transport properties behave seemingly very differently for the amplifying and the absorbing chains, the logarithmic resistance $u = \ln(1 + R_4)$ where R_4 is the 4-probe resistance gives a unified description of both the cases. It is found that the ensemble-averaged $\langle u \rangle$ increases linearly with length indicating exponential growth of resistance. While in contrast to the case of Anderson localization (random hermitean matrix), the variance of u could be orders of magnitude smaller in the non-hermitean case, the distribution of u still remains non-Gaussian even in the large length limit.

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*e-mail address: asok@hp2.saha.ernet.in

The study of interference of waves multiply scattered from a system of scatterers with non-hermitean hamiltonians has of late become very fashionable. There are two classes of problems in this respect; one in which the non-hermiticity is in the nonlocal part [1, 2] of the hamiltonian and the other in which the non-hermiticity is in the local part (typically in one-body potentials) [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. The first category is understood to represent, among other things, the physics of vortex lines pinned by columnar defects where the depinning is achieved [1] by a sufficiently high transverse magnetic field (represented by an imaginary vector potential). In the second category, it is well-known that an imaginary term in the local part of the non-hermitean hamiltonian behaves like a source or a sink (depending on the sign). We will be concerned here with non-hermiticity in the one-body potential only. In a disordered chain with random but real-valued site-potentials, almost all the states are exponentially localized and hence an incident wave ($\sim e^{ikx}$) propagating in the positive x -direction is completely backscattered due to the well-known localization effects [17]. On the other hand, a purely ordered chain with fixed absorbing site-potentials (sinks for particles) is intuitively expected to lead to exponential decay of the transmittance (forward-scattering), and so it does. While similar intuition may lead to the expectation that the transmittance would increase indefinitely if each of the fixed imaginary site-potentials is amplifying (source of particles), in actuality it does not. In a recent paper [9] (referred to as I from now on), it was shown by the author that the sample with amplifying sites do also behave as a perfect reflector in the large length limit, because the perfect lead boundary conditions at the two ends of the chain imposes a real energy spectrum for the non-hermitean matrix [18]. In this work, we generalize over our work in I and consider the effects of disorder on the coherent amplification/ absorption at each site and look for an unified description for both the cases.

We consider a quantum chain of N lattice points (lattice constant unity), represented by the standard single band, tight binding equation:

$$(E - \epsilon_n)c_n = V(c_{n-1} + c_{n+1}). \quad (1)$$

This *open quantum system* is coupled to the external world (two reservoirs at very slightly different electrochemical potentials) with two identical semi-infinite perfect leads on either side. Here E is the fermionic energy, V is the constant nearest neighbor hopping term which is the same in both the leads and the sample, ϵ_n is the site-energy, and c_n is the site amplitude at the n th site. Without any loss of generality, we choose $\epsilon_n = 0$ in the leads and $V = 1$ to set the energy scale. Inside the sample, we choose $\epsilon_n = \epsilon_r + i\eta$ where the real part ϵ_r is random and the imaginary part η is a fixed real number which may be either positive or negative, and $i = \sqrt{-1}$. The random ϵ_r is obtained from an uniform distribution on $[-W/2, W/2]$. Since for an isolated scattering potential with a positive (negative) η the wave-vector (k) has a positive (negative) imaginary part, the wave ($\sim e^{ikx}$) decays (grows) exponentially with x . Thus a medium with positive η at each site is called an absorbing medium and a medium with negative η an amplifying medium. The physical reason for such a description lies in the fact that the scattering in any real medium is never perfectly elastic and that in many cases the deviation from perfectly elastic scattering may be described by absorption through other inelastic channels or amplification due to enhancement of the wave-amplitude (e.g., population inversion in an active medium) of incident particles or waves. The complex transmission amplitude in the ordered case was calculated to be

$$t_A = \frac{(e^{ik_s}e^{-\gamma} - e^{-ik_s}e^{\gamma})(e^{ik} - e^{-ik})e^{-ik(L+2)}}{de^{-ik_sL}e^{\gamma L} - ce^{ik_sL}e^{-\gamma L}}, \quad (2)$$

where

$$c = (e^{ik_s - ik}e^{-\gamma} - 1)^2, d = (e^{-ik_s - ik}e^{\gamma} - 1)^2, \quad (3)$$

and the decay length $1/|\gamma| = l_a$ and the wave-vector k_s are given by

$$E = 2\cos k = (e^{\gamma} + e^{-\gamma})\cos k_s, \quad (4)$$

and

$$\eta = (e^{\gamma} - e^{-\gamma})\sin k_s. \quad (5)$$

The transmittance or the two-probe conductance $T = g_2 = |t_A|^2$ obtained from the Eq.(2) is found to decay monotonically (exponentially) towards zero for a set of absorbers ($\eta > 0$). But, for a set of amplifiers ($\eta < 0$), g_2 increases first to a high value but eventually (for large L) decays as $t_A \sim e^{-|\gamma|L}$. Disorder (in ϵ_r) in 1D is known to give rise to an exponential decay. Thus in the presence of both disorder and absorption/ amplification, one expects the conductance to behave for $L \rightarrow \infty$ as

$$t \sim t_A e^{-\kappa L} \sim e^{-(|\gamma|+\kappa)L}, \quad (6)$$

where $1/\kappa = \xi_d$ is the localization length for the disorder (in the real part) problem. Thus the asymptotic behavior of the transmittance for both the absorbing and the amplifying case is the same (i.e., exponential decay) and the effective localization length ξ for the disordered complex site-potential case is given by $\xi^{-1} = l_a^{-1} + \xi_d^{-1}$. This result has been checked here by using a numerical transfer matrix method [19] both for the amplifying and the absorbing case. The difference in the behavior of T between the two cases (sharp decay as opposed to an initial rise with large oscillations) appears only upto $L \sim \xi$, and disappears for $L \gg \xi$. It was shown in I for the ordered case with $\xi_d = \infty$, i.e., $\xi = l_a$. This may be explained as follows. For concreteness, let us consider an E close to zero (band centre). Then $|d| > |c|$. Now in the absorbing case $\gamma > 0$ making the exponentially increasing term in the denominator dominating from the beginning and hence T decays without much of an interference (and hence oscillation) from the beginning. On the other hand, in the amplifying case $\gamma < 0$, and hence the exponentially growing term in the denominator cannot dominate until some large enough length scale. Also there is a lot of interference between the growing and the decaying terms in the denominator in this case resulting into an initial rise in T with strong oscillations upto a crossover length $L_c = L_c(E, \eta)$ determined by the approximate equality of the two terms in the denominator of Eq.(2), i.e., by $2|\gamma|L_c \sim \ln|d/c|$. For the amplifying chain in the presence of disorder ($W > 0$), another competition to the amplification peak (T_{max}) appears because of localization effects and $L_c(E, W, \eta) \sim 1/2 \xi \ln|d/c| \rightarrow 0$ as $\xi \rightarrow$

0, i.e., either one or both of W and $|\eta| \rightarrow \infty$.

It may be noted though that the reflectance or the two-probe resistance $R = |r|^2$, where r is the complex reflection amplitude (due to backscattering) behaves differently from the transmittance. The asymptotically constant reflectance is reduced ($R_\infty < 1$) for $\eta > 0$ (absorbing chain), and is amplified ($R_\infty > 1$) for $\eta < 0$ (amplifying chain) by factors depending on the disorder strength (W), $|\eta|$ and E . The behavior (on an average) is very similar to that for the ordered case (see the figures in I). For a fixed E and $\eta < 0$, disorder not only reduces the peak position (L_c) as discussed above, but also reduces the T_{max} as well as the asymptotically amplified value of R_∞ . In Fig.1, we show this for the case of $E=0.1$, $\eta = -0.1$ (the same as in the ordered case of I), and for a disorder strength $W=2.0$. One can check in this case that the $l_a \simeq 20$, and $\xi_d \simeq 25$. Thus the effective localization length should be $\xi = (1/l_a + 1/\xi_d)^{-1} \simeq 11$, and this value matches that obtained from Fig.1. Further it will be noted that the peak $T_{max}=1.3$ appears at $L_c=12$ (compared to the $T_{max} \simeq 2800$ and $L_c = 68 \simeq 3l_a$ in the ordered case), and the amplified backscattering $R_\infty=6.3$ (compared to 1600 in the ordered case).

The issue of the phase distribution of the reflection amplitude r is quite important [8, 16, 20] because the evolution of the phase does have important bearings [20] on the evolution of the resistance, these two quantities being coupled to each other. In the ordered case ($W = 0$), the phase distribution is a δ -function at an angle dependent on E , η (both magnitude and sign) and the length L for a small system. It is independent of L and the sign of η for $L \gg l_a$ (see the expression for r in I). On the other hand for a hermitean disorder case ($\eta = 0$), the phase (ϕ) distribution $P_L(\phi)$ is typically quite non-uniform. The distribution is uniform only in the special case of weak localization ($L \sim \xi_d$). It evolves with length and approaches the stationary limit for $L \gg \xi_d$. This stationary distribution $P(\phi)$ has in general two peaks whose strengths increase (while the separation decreases) with W [20]. They approach a single, narrow peak only in the asymptotic limit of $W \rightarrow \infty$. Obviously in the presence of both disorder and non-hermiticity, there should be a competition between

the broadening effect on $P_L(\phi)$ due to a finite disorder and the δ -function narrowing effect due to the absorption/ amplification. While it is generally accepted that a non-zero η suppresses phase fluctuation [8, 16], we find below that the suppression even in the stationary limit ($L \gg \xi$) is only partial in the sense that the phase fluctuation does not become a δ -function unless $|\eta|$ is very large. Further we find that this stationary distribution is independent of the sign of η (whether amplifying or absorbing chain). In this regard, we first show in Fig. 2 the evolution of $P_L(\phi)$ towards its stationary distribution $P(\phi)$ for the case of $E=1.0$, $W=2.5$, and $|\eta|=0.2$ (i.e., for an amplifying and an absorbing chain). For this choice, $l_a \simeq 10$, and $\xi_d \simeq 12$ (in lattice units). Thus the competition between the two effects should be quite strong. In Figs. 2(a) to 2(d), we show the evolution of $P_L(\phi)$ for $L = 3, 7, 20$ and 200 respectively. In all cases the histograms were obtained from 10,000 configurations. In the Fig. 2(d), $L = 200 \gg l_a$ or ξ_d , and hence all the distributions have become stationary (no further change), and as one may note there is no difference between the distributions for $\eta = +0.2$ or -0.2 in this limit. Next in Figs. 3(a)-3(d), we show the stationary distribution $P(\phi)$ as $|\eta|$ is increased from 0.5 to 3.0. The quite broad stationary distribution in Fig.2(d) for $\eta = 0.2$ gets continually shrunk at larger values of $|\eta|$. But even for a large $|\eta| = 3.0$ in Fig.3(d) where the decay length scale $l_a \simeq 0.83$ (less than a lattice unit), the stationary distribution $P(\phi)$ is far from a δ -function. In contrast to a recent work [16], the phase distribution is nowhere even approximately Gaussian (particularly because of two peaks in general). Further our results differ from another recent work [8] in that $P(\phi)$ does not break apart into two narrow peaks as $|\eta| \rightarrow \infty$.

Since the two-probe properties behave differently for absorbing and amplifying chains and our main purpose here is to look for an unified description, we study the evolution of the four-probe resistance $R_4 = R/T$. Another serious reason for doing this is that the probability distribution of T or R for the amplifying chains is not well-behaved in the sense that all the moments of T or R (including the first one, e.g., $\langle T \rangle$) diverge even for a finite L [12, 13]. Since disorder plays an important

role, we look at the logarithmic resistance in the form of $u(L) = \ln(1 + R_4)$. We note that if Ab is the absorption coefficient then $R + T + Ab = 1$, and if Am is the amplification coefficient then $R + T - 1 = Am$. Thus while the quantity u is nothing but the negative of the logarithmic transmittance $-\ln T$ in the hermitean disorder case, $u = \ln(1 - Ab) - \ln T$ in the absorbing case and $u = \ln(1 + Am) - \ln T$ in the amplifying case. For our calculations, we choose the same parameters as in Figs.2 and 3, and show in Fig.4 the evolutions of $u(L)$ and $var(u) = \langle u^2 \rangle - \langle u \rangle^2$ in double-logarithmic plots. In the Fig.4(a) for a hermitean case ($\eta = 0$), both the $\langle u \rangle$ and $var(u)$ diverges algebraically but with different exponents in the regime $0 < L < L_2$ (where $L_2 \sim \xi_d$) in accordance with a recent proposal by the author [20] regarding the existence of a two-parameter scaling in that regime and that in the strong localization limit $L \gg \xi_d$, they diverge with a single exponent (unity) in accordance with the one-parameter scaling theory. In Figs.4(b)-4(d), we introduce increasing amounts of non-hermiticity $\eta = \pm(0.2 - 3.0)$ and find some profound changes in the relative behavior of the moments. The first thing to note is that the moments of u evolve with L almost identically for both the absorbing and the amplifying cases; the difference cannot be shown in the scale of the figure. This already gives the hint that u is the right variable to look at for an unified description of both of these cases. In contrast to the two-probe properties, these moments of u do not diverge and there are no odd-even (in L) oscillations. Further since there are two length scales involved in this problem, one sees a crossover in Fig.4(b) for $|\eta| = 0.2$ from absorption/ amplification dominated (for $L \leq l_a = 10$ in this example) growth in R_4 to the growth affected by localization effects as well (for $L \geq \xi_d = 12$). For larger $|\eta|$ this crossover is still there at smaller lengths and may not be clearly discernible. Further, in contrast to the hermitean disorder case where $var(u)$ crosses $\langle u \rangle$ from the lower side (mildly fluctuating, weakly localized) to the higher side (fluctuations dominated, strongly localized) in the thermodynamic limit, the asymptotic $var(u)$ in the non-hermitean case is always less than $\langle u \rangle$. Thus the fluctuations in u are getting suppressed by coherent amplification/ absorption

and this may be interpreted as a *destruction* of the effect of localization on $var(u)$ (but not on $\langle u \rangle$). Indeed for a large $|\eta|$, $var(u)$ may be orders of magnitude smaller than $\langle u \rangle$. An example is the case of Fig.4(d) for $|\eta|=3.0$, where $var(u)$ is about 2.5 orders of magnitude smaller than $\langle u \rangle$. It clearly demonstrates how strongly the *coherent* amplification/ absorption works against the localization effects in destroying the fluctuations in transport. Yet at the same time, we must notice that this destruction is never strong enough to bring back the self-averaging character (or, diffusive behavior) to the resistance R_4 . Since this destruction mechanism is an altogether coherent effect, it cannot destroy the the exponential growth of the mean and the variance as an incoherent effect (say, due to phonons) would have done.

To check if $P_L(u)$ is at least asymptotically normal (i.e., $P(R_4)$ log-normal), one may calculate the kurtosis parameter $K = 1/2(3 - c_4/c_2^2)$ where $c_2 = var(u)$ and $c_4 = (u - \langle u \rangle)^4$ at each L . For a normal distribution, $c_4 = 3c_2^2$ and hence $K = 0$. In Fig.5, we plot $K(L)$ as a function of the system size L for the parameters of Fig.4(b). It may not be surprising that $K(L)$ is not close to zero ($P_L(u)$ far from normal) for small lengths, but for much larger lengths $L \gg 10\xi$ (where $\xi \simeq 5$), $K(L)$ still fluctuates within $[+0.1, -0.1]$. Thus u does not seem to be asymptotically normal, and hence R_4 does not seem to be asymptotically log-normal.

To summarize, we have looked at the effect of backscattering in a disordered chain with coherently amplifying or absorbing site-potentials. Disorder enhances the exponential decay of transmittance in the presence of absorbers, while it suppresses the transmittance peak (and the amplified asymptotic reflectance) due to the amplifiers. The stationary ($L \rightarrow \infty$) probability density of the phase of the reflection amplitude is generally an asymmetric double-peaked function (not even approximately Gaussian) whose width is progressively suppressed towards zero (δ -function-like form as in the case of a pure amplifying/ absorbing chain) by larger amplifying/ absorbing strength $|\eta|$ or by larger disorder. The four-probe resistance R_4 gives an unifying description for both the amplifying and the absorbing chain, and unlike in the case of Anderson localization the amplifying/ absorbing effect can

suppress the fluctuations of the logarithmic resistance by orders of magnitude compared to the average logarithmic resistance. Study of the kurtosis indicates that the probability density of the resistance R_4 is not asymptotically log-normal.

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Figure Captions:

Fig.1 Evolution of the averages of the logarithmic reflectance ($\langle \ln R \rangle$) and the logarithmic transmittance ($\langle \ln T \rangle$) as a function of chain length L , for a coherently amplifying chain with $\eta = -0.1$, Fermi energy $E = 0.1$, and a disorder strength $W = 2.0$. The averages are calculated at each L for 10000 configurations.

Fig.2 Evolution of the distribution of the phase $P_L(\phi)$ with L for both an amplifying and an absorbing chain $\eta = \pm 0.2$, $E = 1.0$, $W = 2.5$. Histograms representing the distribution are drawn using 10000 configurations each for (a) $L = 3$, (b) $L = 7$, (c) $L = 20$, and (d) $L = 200$. The distributions for the pure amplifying/ absorbing as well as the hermitean ($\eta = 0$) disorder cases are also shown for each L . The Fig.2(d) represents the stationary distribution $P(\phi)$.

Fig.3 Stationary phase distribution $P(\phi)$ for $E = 1.0$ and $W = 2.5$ using $L = 200$ for (a) $|\eta| = 0.5$, (b) $|\eta| = 1.0$, (c) $|\eta| = 2.0$ and (d) $|\eta| = 3.0$.

Fig.4 Evolution of the mean and variance of $u = \ln(1 + R_4)$, where R_4 is the four-probe resistance, as a function of L . Both amplifying and absorbing chains with 6000 configurations were used, no distinction between them can be made to the scale of the figure. The parameters used are $E = 1.0$, $W = 2.5$, and (a) $|\eta| = 0.0$, (b) $|\eta| = 0.2$, (c) $|\eta| = 1.0$ and (d) $|\eta| = 3.0$.

Fig.5 The dimensionless kurtosis (see text) parameter as a function of length L using $E = 1.0$, $W = 2.5$, $|\eta| = 0.2$ and 6000 configurations.

Fig.1: $E=0.1$, $W=2.0$, $\eta=-0.1$, 10000 config.

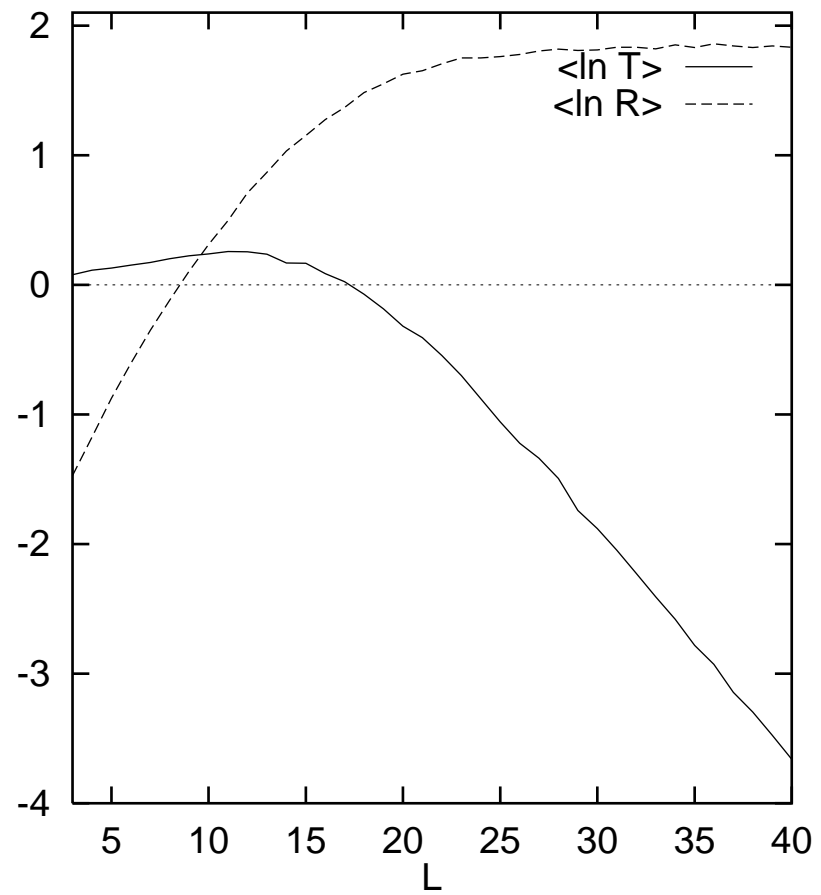


Fig.2a: Dist. of phase, $L=3$, $E=1.0$, 10000 config.

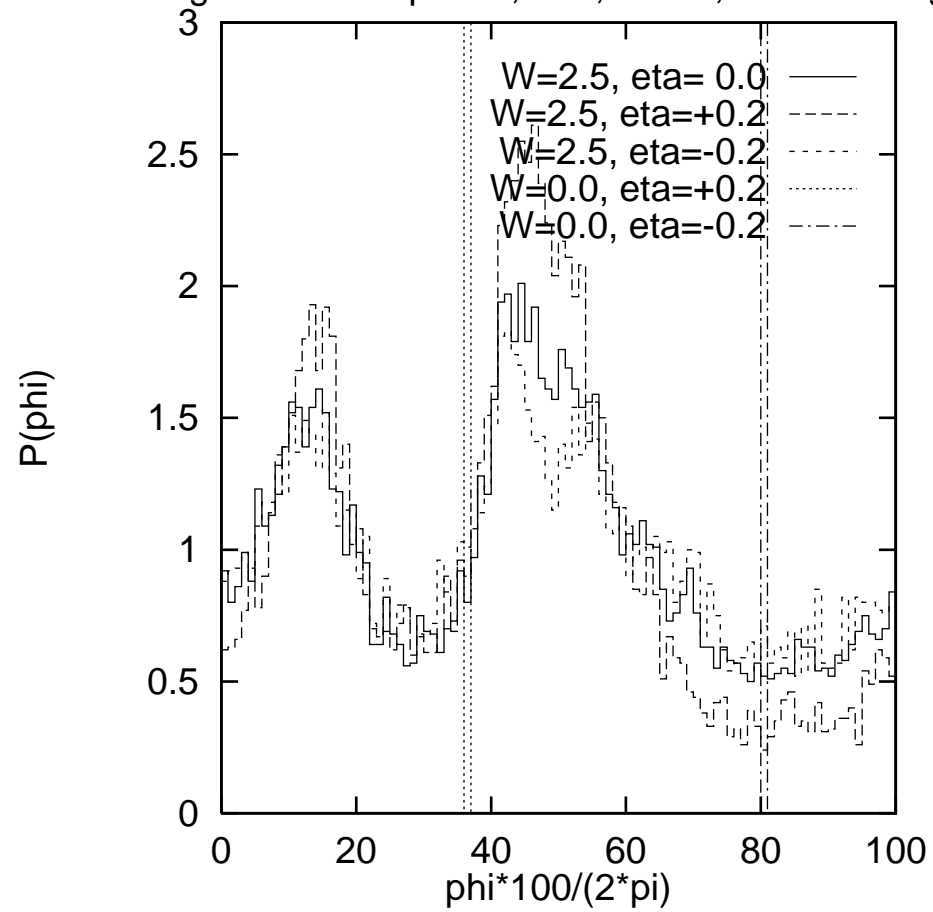


Fig.2b: Dist. of phase, $L=7$, $E=1.0$, 10000 config.

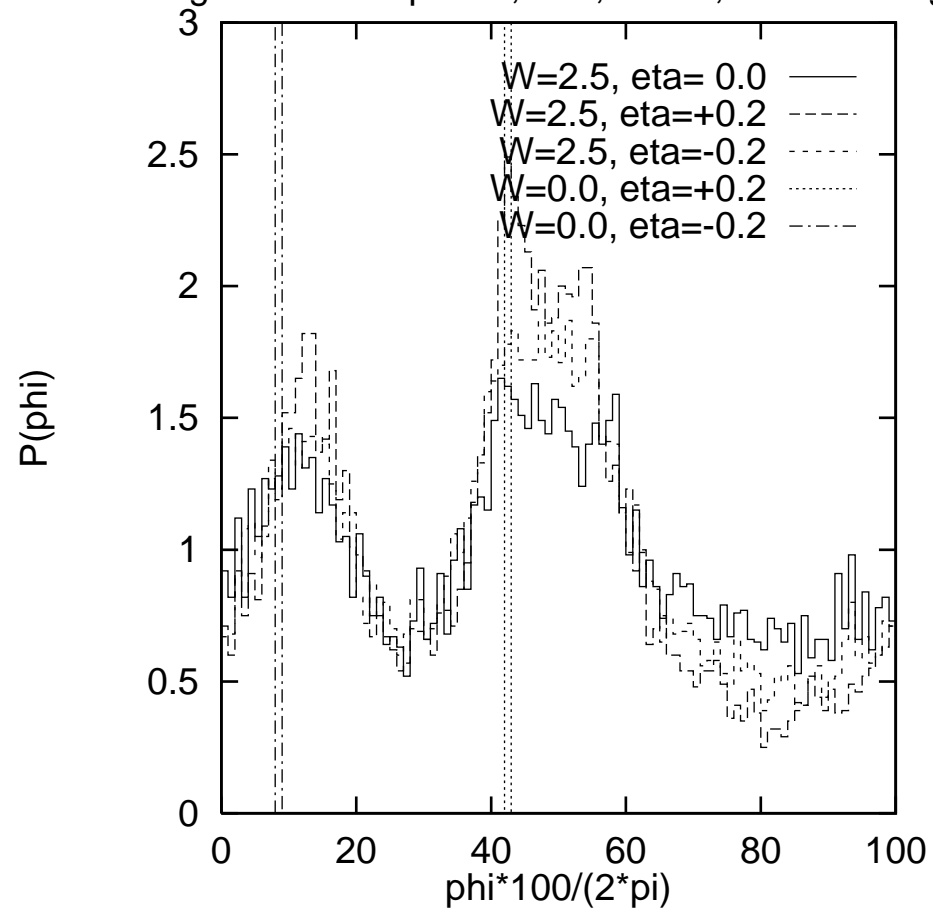


Fig.2c: Dist. of phase, $L=20$, $E=1.0$, 10000 config.

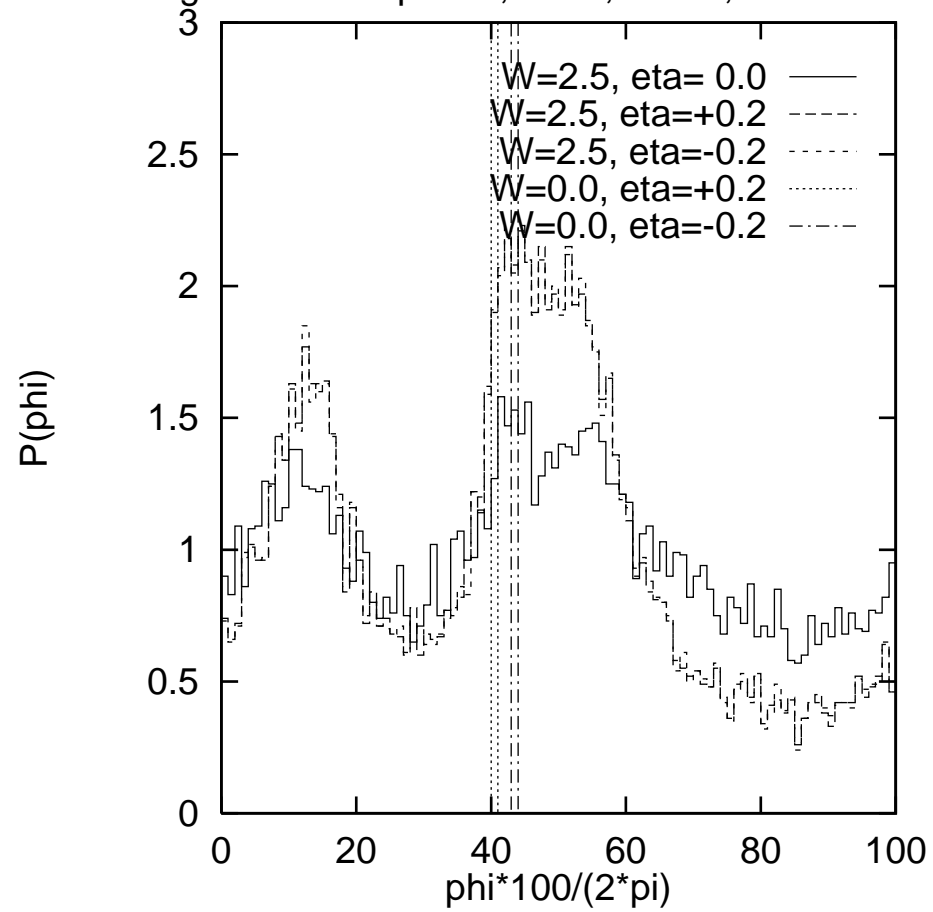


Fig.2d: Dist. of phase, $L=200$, $E=1.0$, 10000 config.

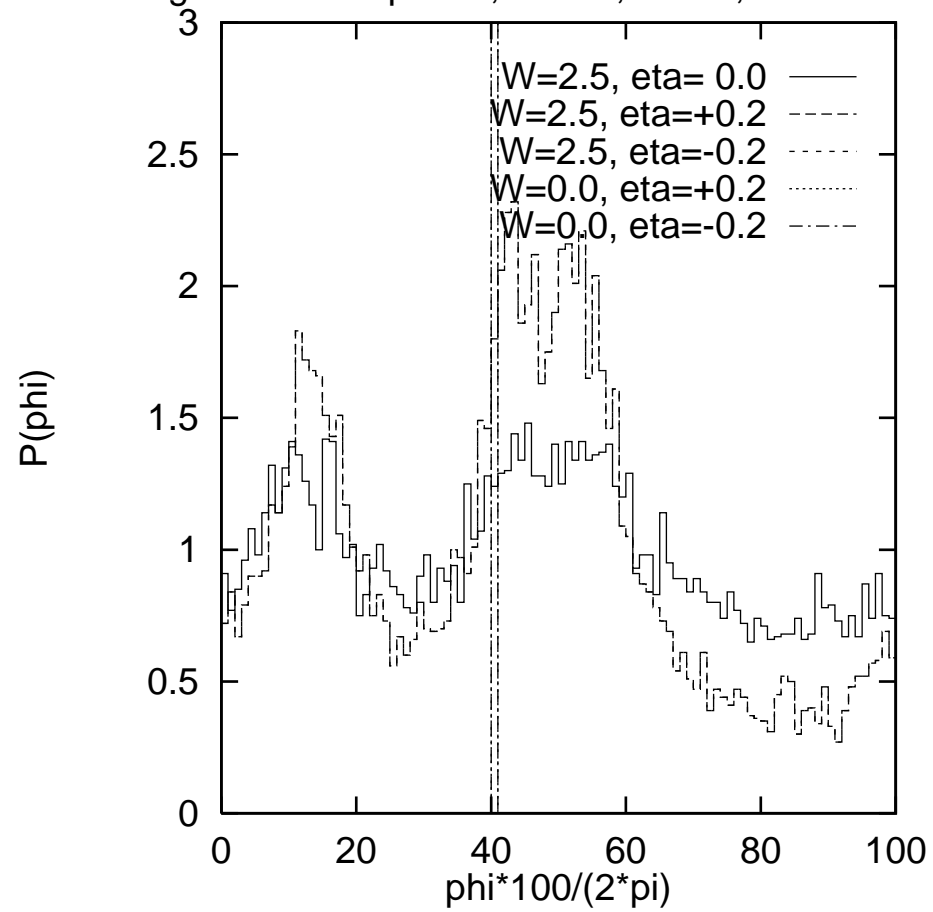


Fig.3a: Stn. Phase Dist., $L=200$, $E=1.0$, 10000 config.

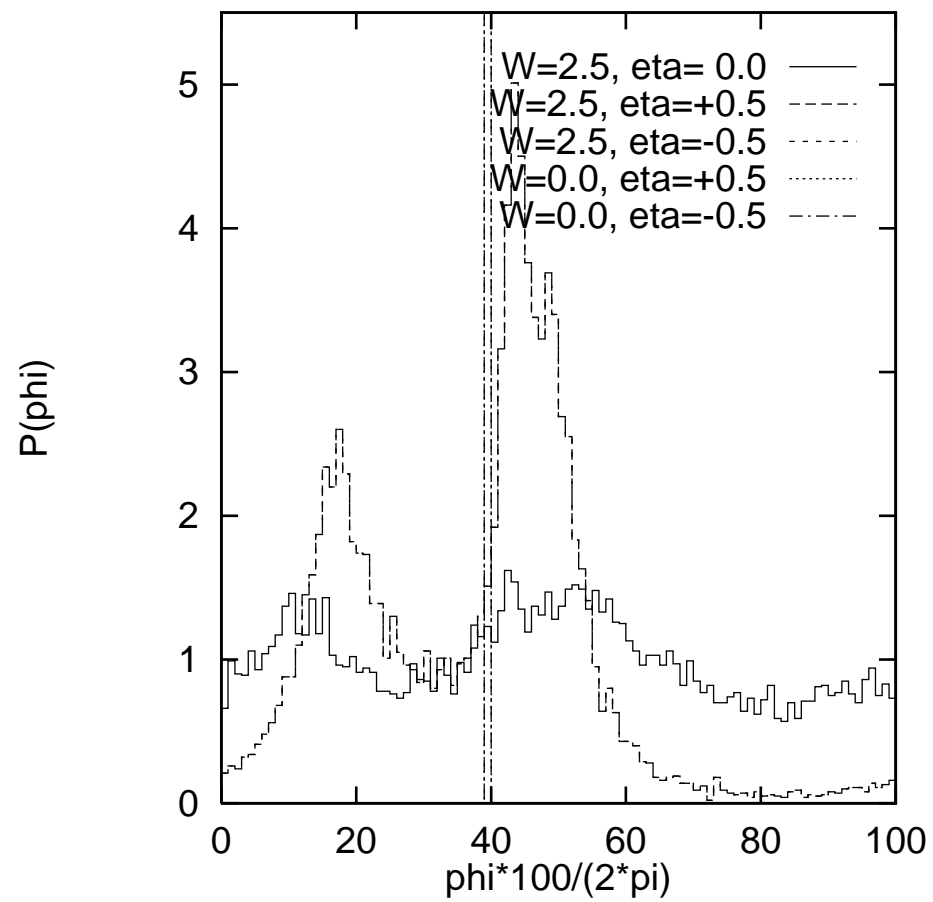


Fig.3b: Stn. Phase Dist., $L=200$, $E=1.0$, 10000 config.

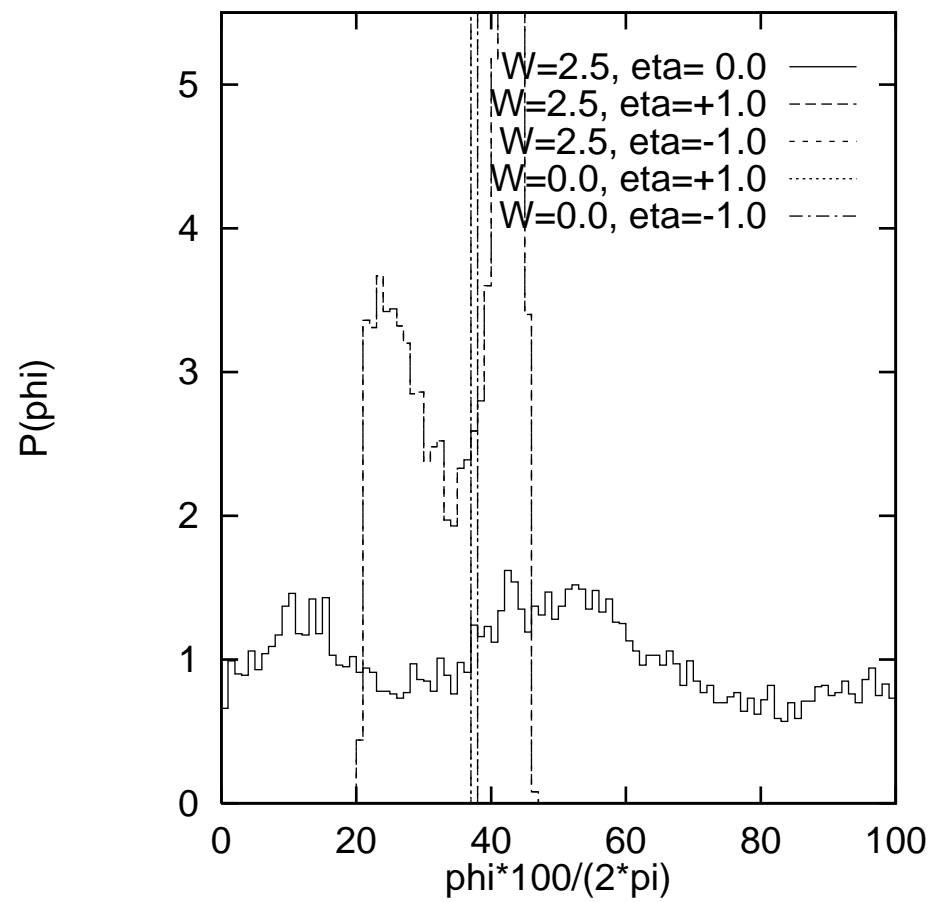


Fig.3c: Stn. Phase Dist., $L=200$, $E=1.0$, 10000 config.

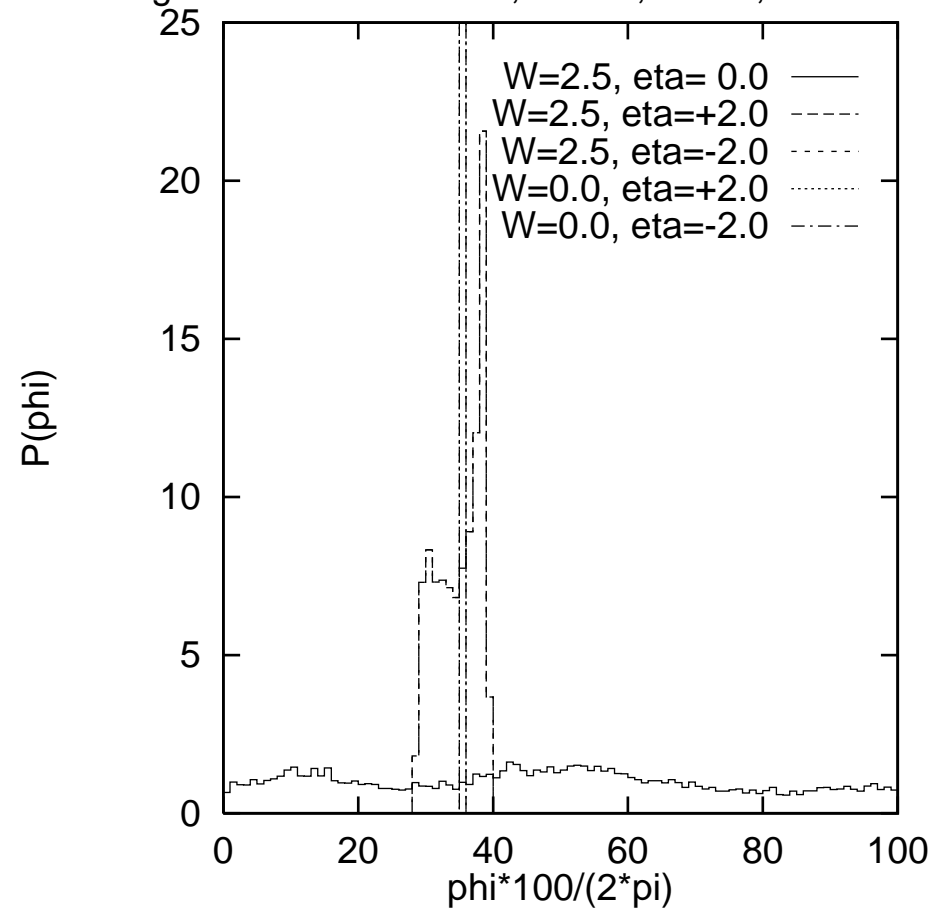


Fig.3d: Stn. Phase Dist., L=200, E=1.0, 10000 config.

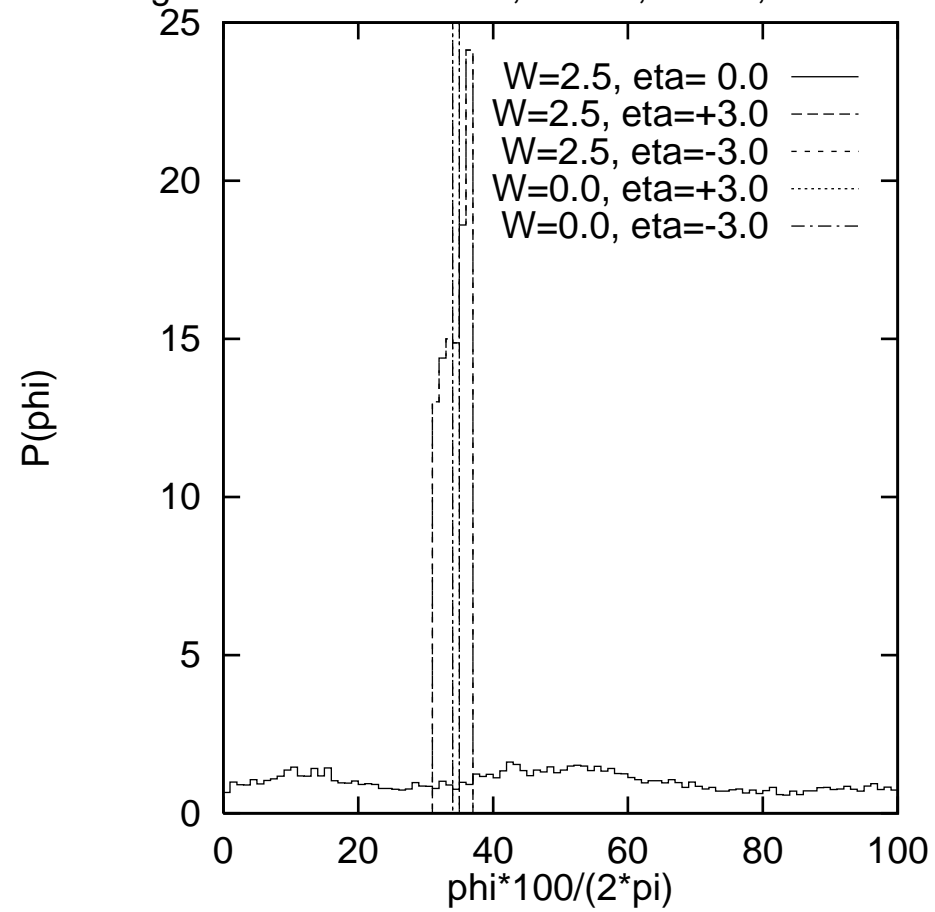


Fig.4a: $E=1.0$, $W=2.5$, $\eta=0.0$, 6000 config.

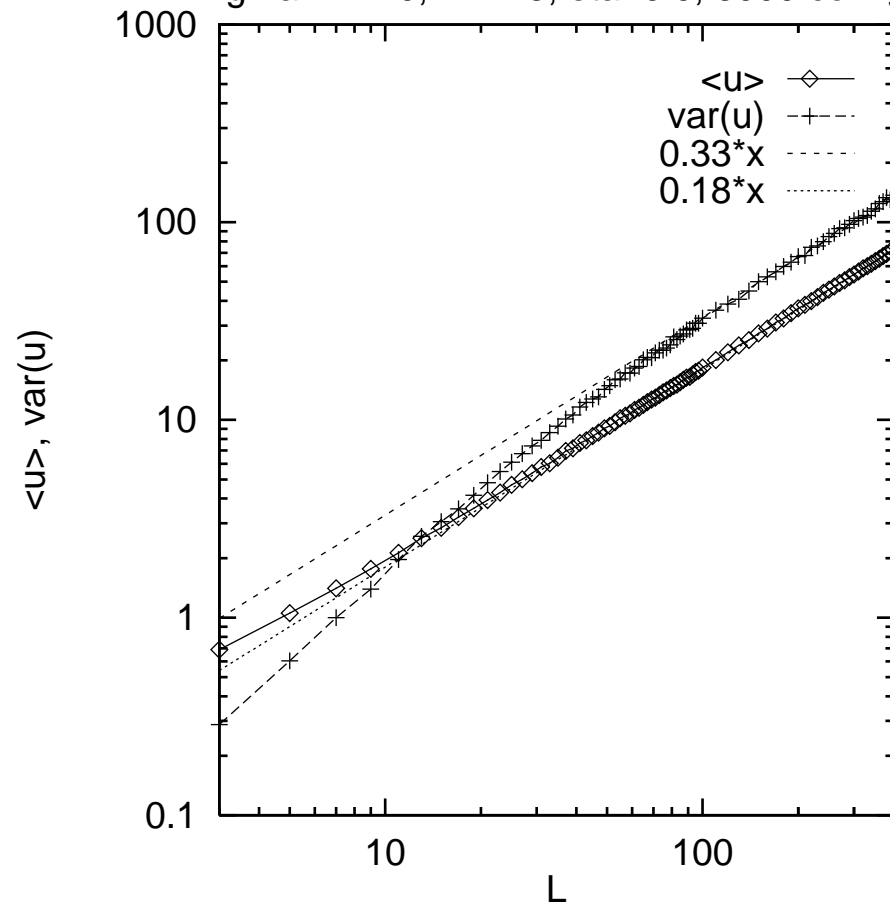


Fig.4b: $E=1.0$, $W=2.5$, $|\eta|=0.2$, 6000 config.

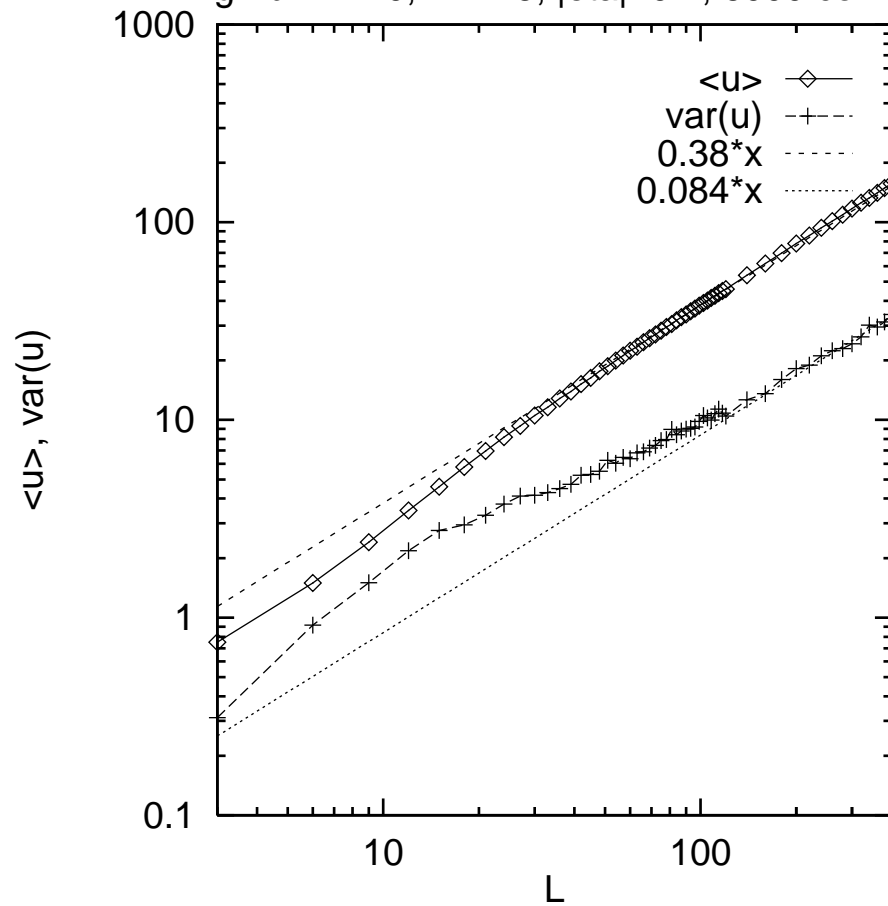


Fig.4c: $E=1.0$, $W=2.5$, $|\eta|=1.0$, 6000 config.

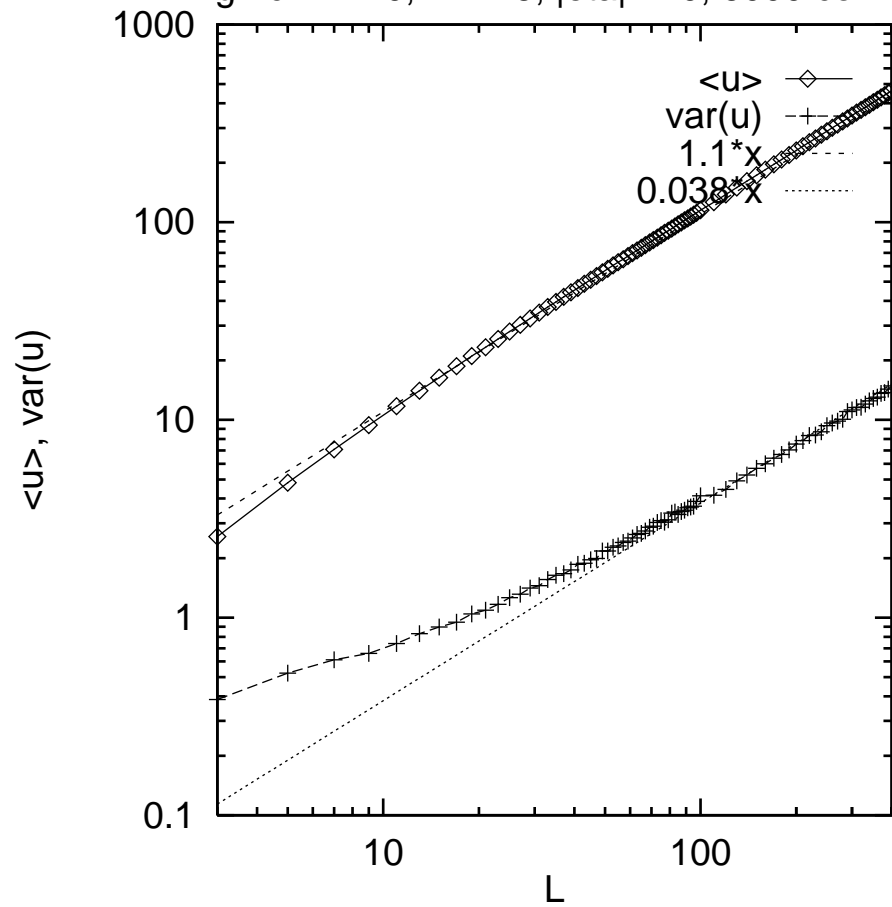


Fig.4d: $E=1.0$, $W=2.5$, $|\eta|=3.0$, 6000 config.

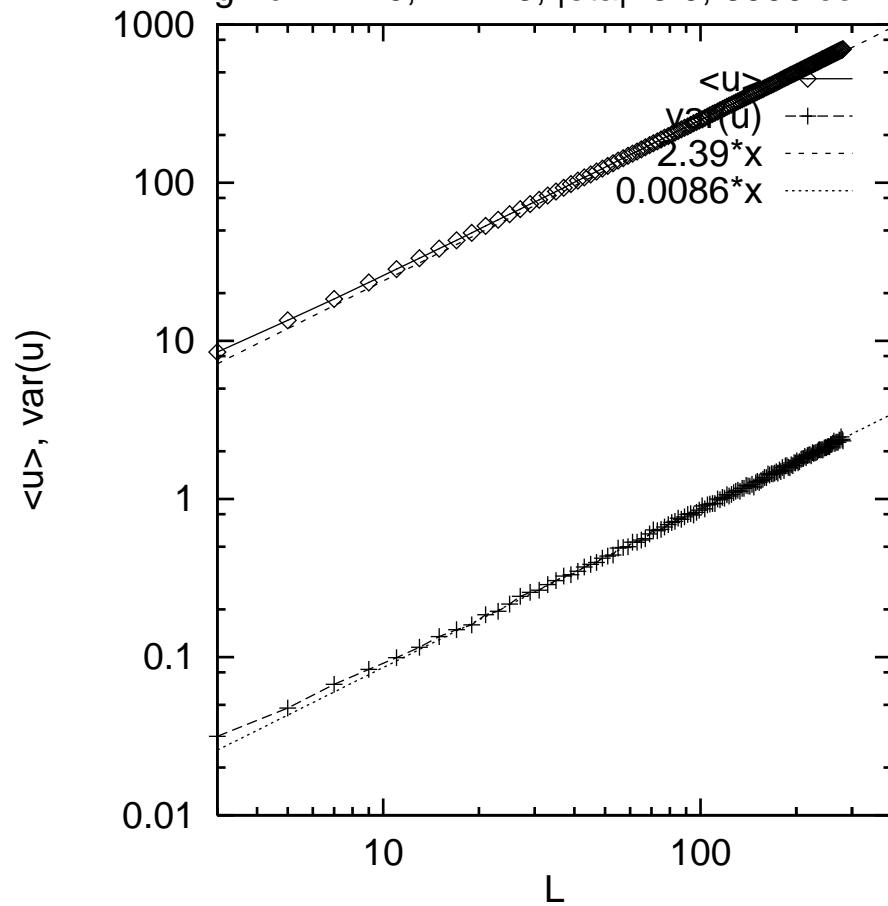


Fig.5: Kurtosis; $E=1.0$, $W=2.5$, $|\eta|=0.2$, 6000 config.

